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Gamma and sine functions for Lie groups and period integrals

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ABSTRACT

We study gamma and sine functions attached to a finite dimensional linear representation of a Lie group including their multiple versions via zeta regularizations. These definitions allow us to understand several multiple gamma and sine functions known so far, such as Barnes' gamma functions, Shintani's sine functions, and their q-analogues, in a unified way. Further, we discuss period integrals for such gamma functions.

1. INTRODUCTION

The multiple gamma function is first studied by Barnes [2] in 1900. Since then, several multiple gamma functions and multiple sine functions have been introduced, and have played important roles in number theory [30] and [14–16]. Also, these functions are closely connected with zeta functions theory such as developed in [22]. Furthermore, the zeta regularization method has been developed with various trial for obtaining determinant expressions of zeta functions and L-functions (see, e.g., [5,20] and [13]). It is indeed an important subject from the viewpoint of special functions, especially, in spectral geometry and mathematical physics (see, e.g., [27, 28,8,11,20]).

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The purpose of the present paper is to study a gamma and sine functions attached to a finite dimensional linear representation of a Lie group via zeta regularizations. In order to define such gamma functions, it is necessary to study the analytic continuation of the zeta functions attached to a representation of a group [32]. This new notion naturally involves multiple versions of gamma and sine functions. These definitions, therefore, allow us to understand several multiple gamma and sine functions known so far, such as Barnes' gamma functions, Shintani's sine functions, and their q-analogues in a unified way. For instance, one can show that the multiple elliptic gamma function [26] (see also [25]) is regarded as a multiple sine function defined from a q-analogue of the Barnes multiple gamma function (which is a simple case of a q-Shintani sine function defined in Section 3 of [32]). Moreover, we generalize the Raabe integral formula $\int_0^1 \log\Gamma(t)\,dt = \frac{1}{2}\log(2\pi)$ to some such gamma functions in Section 4. We note that, quite recently, the Raabe type formula for the Shintani zeta function is deeply studied in [9] (see also [4,6]).

Let G be a Lie group. Let (ρ, V) be a finite dimensional (linear) representation of G. For a r-tuple of elements $\underline{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_r)$ and $g \in \mathrm{End}(V)$, we define a multiple gamma function $\Gamma^{(r)}_{G,\rho}(g; \underline{\gamma})$ via the zeta regularized product as

$$(1.1)\qquad \Gamma^{(r)}_{G,\rho}(g;\underline{\gamma})^{-1} := \prod_{\underline{\mathbf{n}}\geqslant\underline{0}} \mathrm{tr}\big(\rho(\underline{\gamma}^{\underline{\mathbf{n}}})g\big) = \prod_{n_1,n_2,\dots,n_r=0}^{\infty} \mathrm{tr}\big(\rho(\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r})g\big)$$

where $\underline{\gamma}^{\underline{\mathbf{n}}} = \gamma_1^{n_1}\cdots\gamma_r^{n_r}$ for $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$ and $\underline{\mathbf{n}} = (n_1, \dots, n_r)$, when the associated zeta function has good properties explained below. The regularized product $\prod$ is defined as follows (see, e.g., [11,19,13]): For a sequence $\mathbf{a} = \{a_n\}_{n=0,1,\dots}$ of non-zero complex numbers, we define a zeta function attached to $\mathbf{a}$ by the Dirichlet series $\zeta_{\mathbf{a}}(s) = \sum_{n=0}^{\infty} a_n^{-s}$. We assume that the series converges absolutely when $\mathrm{Re}\, s$ is large enough and further that the zeta function $\zeta_{\mathbf{a}}(s)$ can be meromorphically continued to the region containing $s = 0$. Thus, in particular, we assume that the order of each $\rho(\gamma_i)$ is not finite. Then we define the (dot-)regularized product of the sequence $\mathbf{a}$ by

$$(1.2)\qquad \prod_{n=0}^{\infty} a_n = \exp\left(-\operatorname*{Res}_{s=0} \frac{\zeta_{\mathbf{a}}(s)}{s^2}\right).$$

Hence, if we define the zeta function $\zeta^{(r)}_{G,\rho}(s, g; \underline{\gamma})$ of the corresponding data by the Dirichlet series $\zeta^{(r)}_{G,\rho}(s, g; \underline{\gamma}) := \sum_{\underline{\mathbf{n}}\geqslant\underline{0}} \mathrm{tr}(g\rho(\underline{\gamma}^{\underline{\mathbf{n}}}))^{-s}$, which converges absolutely if $\mathrm{Re}\, s$ is sufficiently large, and assume that $\zeta^{(r)}_{G,\rho}(s, g; \underline{\gamma})$ can be meromorphically extended to the origin $s = 0$ we have

$$(1.3)\qquad \Gamma^{(r)}_{G,\rho}(g;\underline{\gamma}) = \exp\Big(\operatorname*{Res}_{s=0} \zeta^{(r)}_{G,\rho}(s, g; \underline{\gamma})/s^2\Big).$$

For instance, let $(\rho, \mathbb{C}^2)$ be a 2-dimensional representation of the additive group $G = \mathbb{C}$ defined by $\rho(z) := \left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}\right) \in SL_2(\mathbb{C})$. Then, for $\gamma = 1$ and $g = \left(\begin{smallmatrix}x-1 & 0\\ 1 & 1\end{smallmatrix}\right) \in$

$\mathrm{End}(\mathbb{C}^2)$, we have $\mathrm{tr}(\rho(1)^n g) = x + n$, whence $\Gamma_{\mathbb{C},\rho}(g;\gamma) := \boxed{\cdot}\!\!\prod_{n=0}^{\infty}(n+x)^{-1} = \Gamma(x)/\sqrt{2\pi}$ by the Lerch formula [24] for the classical gamma function. Similarly, one can show that the Barnes multiple gamma function $\Gamma_r(x) = \exp(\zeta_r'(0,x))$ defined through the multiple Hurwitz zeta function $\zeta_r(s,x) = \sum_{n_1,n_2,\ldots,n_r=0}^{\infty}(n_1 + \cdots + n_r + x)^{-s}$ is obtained in our scheme. (See Example 3.3. In general, see [32] for the Shintani zeta function.) Moreover, in view of the reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ of the $\Gamma(x)$, we may naturally define a multiple sine function, or rather, it is necessary to introduce both a right, left and central multiple sine functions because of the non-commutativity of γ_j's (see Section 3).

Throughout the paper, for simplicity, we are assuming that no element $\rho(\gamma_j) \in GL(V)$ is of finite order. Actually, if we consider the case where some $\rho(\gamma_j)$ is an element of finite order N_j then the product taken over only from 0 to $N_j - 1$. We fix the log branch as

$$\log z = \log|z| + i \arg z \quad (-\pi \leqslant \arg z < \pi).$$

Also, when the zeta function $\zeta_{\mathbf{a}}(s)$ is holomorphic at $s = 0$ we occasionally use the notation $\prod$ (according to [5]) in place of $\boxed{\cdot}\!\!\prod$ by dropping the centered dot. Moreover, we sometimes write $\Gamma_{G,\rho}^{(r)}(g)$ instead of $\Gamma_{G,\rho}^{(r)}(g;\underline{\gamma})$ if there is no fear of confusion.

2. GAMMA FUNCTIONS FOR $SL_2(\mathbb{C})$

To help a clear understanding of the picture, in this section, we give an explicit description of the gamma function of rank 1 (for a hyperbolic element γ) attached to an irreducible finite dimensional representation of $G = SL_2(\mathbb{C})$.

For each non-negative integer ℓ, let $\mathbb{C}[x]^\ell$ be the space of polynomials in x degree at most ℓ. Denote $(\rho_\ell, \mathbb{C}[x]^\ell)$ the irreducible $(\ell+1)$-dimensional representation of $G = SL_2(\mathbb{C})$ defined by

$$(2.1) \qquad \big(\rho_\ell(h)p\big)(x) = (cx+d)^\ell p\left(\frac{ax+b}{cx+d}\right) \quad \left(p \in \mathbb{C}[x]^\ell,\ h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right).$$

We determine the gamma functions $\Gamma_{SL_2(\mathbb{C}),\rho_\ell}^{(1)}(g,\gamma)$ for $(\rho_\ell, \mathbb{C}[x]^\ell)$. To describe the result, we introduce the function $G_q^{(\ell)}(t_1, t_2, \ldots, t_\ell)$ defined by

$$(2.2) \qquad G_q^{(\ell)}(t_1, t_2, \ldots, t_\ell) = \prod_{n=0}^{\infty}\big(1 + q^n t_1 + q^{2n} t_2 + \cdots + q^{\ell n} t_\ell\big)$$

for $0 < q < 1$. Obviously, the product converges absolutely. When $\ell = 1$ it is the q-shifted factorial.

Theorem 2.1. *Let $\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in SL_2(\mathbb{C})$. Suppose $|a| > 1$. Let $g = \mathrm{diag}(x_\ell, x_{\ell-2}, \ldots, x_{-\ell}) \in \mathrm{End}(\mathbb{C}[x]^\ell)$. Assume that $x_\ell \neq 0$. Then we have*

$$(2.3) \qquad \Gamma_{SL_2(\mathbb{C}),\rho_\ell}^{(1)}(g,\gamma)^{-1} = a^{-\frac{\ell}{12}} x_\ell^{\frac{\log x_\ell}{2\ell \log a} - \frac{1}{2}} G_{a^{-2}}^{(\ell)}\left(\frac{x_{\ell-2}}{x_\ell}, \frac{x_{\ell-4}}{x_\ell}, \ldots, \frac{x_{-\ell}}{x_\ell}\right).$$

Proof. Since $\rho_\ell(\gamma)x^k = a^{2k-\ell}x^k$ $(0 \leqslant k \leqslant \ell)$, we have

$$\operatorname{tr}\bigl(\rho_\ell(\gamma)^n g\bigr) = \sum_{j=0}^{\ell} x_{\ell-2j} a^{n(\ell-2j)}.$$

Denote the corresponding zeta function by $\zeta_\ell(s, g)$. Then

$$\begin{aligned}
\zeta_\ell(s, g) &= \sum_{n=0}^{\infty} \operatorname{tr}\bigl(\rho(\gamma)^n g\bigr)^{-s} = \sum_{n=0}^{\infty} \left[x_\ell a^{n\ell} + \sum_{j=1}^{\ell} x_{\ell-2j} a^{n(\ell-2j)} \right]^{-s} \\
&= x_\ell^{-s} \sum_{n=0}^{\infty} a^{-ns\ell} \left\{ 1 + x_\ell^{-1} a^{-n\ell} \left[\sum_{j=1}^{\ell} x_{\ell-2j} a^{n(\ell-2j)} \right] \right\}^{-s}.
\end{aligned}$$

For simplicity, we assume that $|x_\ell| \gg |x_{\ell-2j}|$ $(1 \leqslant j \leqslant \ell)$. (This assumption can be removed.) Then, since $|x_\ell^{-1} a^{-n\ell}[\sum_{j=1}^{\ell} x_{\ell-2j} a^{n(\ell-2j)}]| < 1$ it follows from the binomial theorem that

$$\begin{aligned}
\zeta_\ell(s, g) &= x_\ell^{-s} \sum_{n=0}^{\infty} a^{-ns\ell} \sum_{k=0}^{\infty} \binom{-s}{k} x_\ell^{-k} a^{-kn\ell} \left[\sum_{j=1}^{\ell} x_{\ell-2j} a^{n(\ell-2j)} \right]^k \\
&= x_\ell^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} x_\ell^{-k} \sum_{n=0}^{\infty} a^{-ns\ell} \left[\sum_{j=1}^{\ell} x_{\ell-2j} a^{-2nj} \right]^k.
\end{aligned}$$

Recall the multinomial theorem

$$(a_1 + \cdots + a_m)^n = \sum_{p_1+\cdots+p_m=n} \frac{k!}{p_1! \cdots p_\ell!} a_1^{p_1} \cdots a_m^{p_m}.$$

Then we have

$$\begin{aligned}
\zeta_\ell(s, g) &= x_\ell^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} x_\ell^{-k} \sum_{p_1+\cdots+p_\ell=k} \frac{k!}{p_1! \cdots p_\ell!} x_{\ell-2}^{p_1} x_{\ell-4}^{p_2} \cdots x_{-\ell}^{p_\ell} \\
&\quad \times \sum_{n=0}^{\infty} a^{-n\{s\ell+2(p_1+2p_2+\cdots+\ell p_\ell)\}} \\
&= x_\ell^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} x_\ell^{-k} \sum_{p_1+\cdots+p_\ell=k} \frac{k!}{p_1! \cdots p_\ell!} \\
&\quad \times \frac{x_{\ell-2}^{p_1} x_{\ell-4}^{p_2} \cdots x_{-\ell}^{p_\ell}}{1 - a^{-\{s\ell+2(p_1+2p_2+\cdots+\ell p_\ell)\}}}.
\end{aligned}$$

This shows $\zeta_\ell(s, g)$ is meromorphic around $s = 0$. In particular, since $\binom{-s}{k} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}$, we see that

$$\zeta_\ell(s,g)=\frac{x_\ell^{-s}}{1-a^{-s\ell}}+s\sum_{k=1}^{\infty}\frac{(-x_\ell)^{-k}}{k}$$
$$\times\sum_{p_1+\cdots+p_\ell=k}\frac{k!}{p_1!\cdots p_\ell!}\frac{x_{\ell-2}^{p_1}x_{\ell-4}^{p_2}\cdots x_{-\ell}^{p_\ell}}{1-a^{-2(p_1+2p_2+\cdots+\ell p_\ell)}}+O(s^2).$$

Note that

$$\frac{x_\ell^{-s}}{1-a^{-s\ell}}=\frac{1}{\ell\log a}\frac{1}{s}-\frac{\log x_\ell}{\ell\log a}+\frac{1}{2}$$
$$+\left\{\frac{(\log x_\ell)^2}{2\ell\log a}-\frac{1}{2}\log x_\ell+\frac{\ell\log a}{12}\right\}s+O(s^2).$$

Also, from the definition, we see that

$$\log G_{a^{-2}}^{(\ell)}\left(\frac{x_{\ell-2}}{x_\ell},\frac{x_{\ell-4}}{x_\ell},\dots,\frac{x_{-\ell}}{x_\ell}\right)$$
$$=\sum_{n=0}^{\infty}\log\left\{1+x_\ell^{-1}\left(a^{-2n}x_{\ell-2}+a^{-4n}x_{\ell-4}+\cdots+a^{-2\ell n}x_{-\ell}\right)\right\}$$
$$=-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}x_\ell^{-k}\left(a^{-2n}x_{\ell-2}+a^{-4n}x_{\ell-4}+\cdots+a^{-2\ell n}x_{-\ell}\right)^k$$
$$=-\sum_{k=1}^{\infty}\frac{(-x_\ell)^{-k}}{k}\sum_{p_1+\cdots+p_\ell=k}\frac{k!}{p_1!\cdots p_\ell!}\frac{x_{\ell-2}^{p_1}x_{\ell-4}^{p_2}\cdots x_{-\ell}^{p_\ell}}{1-a^{-2(p_1+2p_2+\cdots+\ell p_\ell)}}.$$

Therefore, we obtain

$$\operatorname*{Res}_{s=0}\frac{\zeta_\ell(s,g)}{s^2}=\frac{(\log x_\ell)^2}{2\ell\log a}-\frac{1}{2}\log x_\ell+\frac{\ell\log a}{12}$$
$$-\log G_{a^{-2}}^{(\ell)}\left(\frac{x_{\ell-2}}{x_\ell},\frac{x_{\ell-4}}{x_\ell},\dots,\frac{x_{-\ell}}{x_\ell}\right).$$

Hence the result follows immediately. □

Remark 1. By a quite similar way, one can determine the gamma function $\Gamma^{(r)}_{SL_2(\mathbb{C}),\rho_\ell}(g,\underline{\gamma})$ of rank r attached to the irreducible $(\ell+1)$-dimensional irreducible representation of $SL_2(\mathbb{C})$. Actually, for $0<q_1,\ \dots,\ q_r<1$, if we define a multiple version of $G_q^\ell(t_1,\ \dots,\ t_\ell)$ by

$$G^{(r,\ell)}_{(q_1,\dots,q_r)}(t_1,\dots,t_\ell)=\prod_{n_1,\dots,n_r\geqslant 0}\left(1+q_1^{n_1}\cdots q_r^{n_r}t_1+\cdots+q_1^{\ell n_1}\cdots q_r^{\ell n_r}t_\ell\right),\tag{2.4}$$

since $\operatorname{tr}\rho_\ell(\gamma_1^{n_1}\cdots\gamma_r^{n_r})=\sum_{j=0}^{\ell}x_{\ell-2j}a_1^{n_1(\ell-2j)}\cdots a_r^{n_r(\ell-2j)}$, one can show that (the inverse of) the gamma function $\Gamma^{(r)}_{SL_2(\mathbb{C}),\rho_\ell}(g,\underline{\gamma})$ is essentially given by $G^{(r,\ell)}_{(a_1^{-2},\dots,a_r^{-2})}(\frac{x_{\ell-2}}{x_\ell},\frac{x_{\ell-4}}{x_\ell},\dots,\frac{x_{-\ell}}{x_\ell})$. Details are left to the reader.

We close this section by providing another type of example.

Example 2.1. Let $G = GL_3(\mathbb{C})$ and consider the natural representation of G. Let

$$h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t \\ 0 & t & 0 \end{pmatrix} \quad \big(0 < |t| < 1\big).$$

Then we have $\boxdot_{m=0}^{\infty} \operatorname{tr}(h^m g) = \boxdot_{m=0}^{\infty}(2^m + mt)$. It actually exists. In fact, since

$$\begin{aligned}
\zeta_h(s, g) &= \sum_{m=0}^{\infty} 2^{-ms}(1 + tm2^{-m})^{-s} \\
&= 1 + \sum_{m=1}^{\infty} 2^{-ms} \sum_{\ell=0}^{\infty} (-1)^{\ell} \binom{s+\ell-1}{\ell} \big(tm2^{-m}\big)^{\ell} \\
&= 1 + \frac{1}{2^s - 1} + s \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} \sum_{m=1}^{\infty} (tm)^{\ell} 2^{-m\ell} + \mathrm{O}\big(s^2\big) \\
&= \frac{1}{\log 2} \cdot \frac{1}{s} + \frac{1}{2} + \frac{\log 2}{12} s - s \sum_{m=1}^{\infty} \log(1 + tm2^{-m}) + \mathrm{O}\big(s^2\big),
\end{aligned}$$

by a calculation similar to the one in the preceding theorem, we observe that $\zeta_h(s, g)$ is meromorphic at $s = 0$. Hence it follows that $\Gamma_h(g) := \boxdot_{m=0}^{\infty} \operatorname{tr}(h^m g) = 2^{-\frac{1}{12}} \prod_{m=1}^{\infty}(1 + tm2^{-m})$.

3. FUNCTIONAL EQUATIONS AND SINE FUNCTIONS

Let $g \in GL(V)$. Then, in view of the reflection formula of the gamma function $\Gamma(x)$, we define a left and right sine functions $S_{G,\rho}^{(r,L)}(g; \underline{\gamma})$ and $S_{G,\rho}^{(r,R)}(g; \underline{\gamma})$ respectively by the formulas

$$S_{G,\rho}^{(r,L)}(g; \underline{\gamma}) := \Gamma_{G,\rho}^{(r)}(g; \underline{\gamma})^{-1} \Gamma_{G,\rho}^{(r)}(\rho(\underline{\gamma}) g^{-1}; \underline{\gamma})^{(-1)^r}, \tag{3.1}$$

$$S_{G,\rho}^{(r,R)}(g; \underline{\gamma}) := \Gamma_{G,\rho}^{(r)}(g; \underline{\gamma})^{-1} \Gamma_{G,\rho}^{(r)}(g^{-1} \rho(\underline{\gamma}); \underline{\gamma})^{(-1)^r}. \tag{3.2}$$

In fact, since $\rho(\underline{\gamma})\rho(\underline{\gamma}^{\underline{\mathbf{n}}}) \neq \rho(\underline{\gamma}^{\underline{\mathbf{n}}})\rho(\underline{\gamma})$ in general for $\underline{\mathbf{n}} \in \mathbb{Z}_{>0}^r$, the left sine function is not necessary equal to the right sine function. Furthermore, we define another sine function (which is called a central sine) by

$$S_{G,\rho}^{(r)}(g; \underline{\gamma}) := \boxdot_{\underline{\mathbf{n}} \geqslant \underline{0}} \operatorname{tr}\big(\rho\big(\underline{\gamma}^{\underline{\mathbf{n}}}\big) g\big) \cdot \Big\{ \boxdot_{\underline{\mathbf{n}} \geqslant \underline{0}} \operatorname{tr}\big(\rho\big(\underline{\gamma}^{\underline{\mathbf{n}+1}}\big) g^{-1}\big) \Big\}^{(-1)^{r-1}}, \tag{3.3}$$

where $\underline{\gamma}^{\underline{\mathbf{n}+1}} = \gamma_1^{n_1+1} \cdots \gamma_r^{n_r+1}$. Obviously, if either $r = 1$ or γ_i commutes with γ_j for every pair (i, j) then all the three sine functions coincide; $S_{G,\rho}^{(r,L)}(g; \underline{\gamma}) = S_{G,\rho}^{(r,R)}(g; \underline{\gamma}) = S_{G,\rho}^{(r)}(g; \underline{\gamma})$.

The translation property of the gamma function can be expressed as follows:

Proposition 3.1. *We have*

$$\Gamma^{(1)}_{G,\rho}\big(\rho(\gamma)g;\gamma\big)=\Gamma^{(1)}_{G,\rho}\big(g\rho(\gamma);\gamma\big)=\mathrm{tr}(g)\Gamma^{(1)}_{G,\rho}(g;\gamma). \tag{3.4}$$

Moreover, we have in general

$$\begin{aligned}\Gamma^{(r)}_{G,\rho}\big(\rho(\gamma_j)g;\underline{\gamma}\big)=\Gamma^{(r-1)}_{G,\rho}\big(g;\gamma_1,\dots,\gamma_{j-1},(\gamma_j^{-1}\gamma_{j+1}\gamma_j),\dots,(\gamma_j^{-1}\gamma_r\gamma_j)\big)^{-1}\\ \times\Gamma^{(r)}_{G,\rho}\big(g;\gamma_1,\dots,\gamma_{j-1},\gamma_j,(\gamma_j^{-1}\gamma_{j+1}\gamma_j),\dots,(\gamma_j^{-1}\gamma_r\gamma_j)\big).\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Gamma^{(r)}_{G,\rho}\big(g\rho(\gamma_j);\underline{\gamma}\big)=\Gamma^{(r-1)}_{G,\rho}\big(g;(\gamma_j\gamma_1\gamma_j^{-1}),\dots,(\gamma_j\gamma_{j-1}\gamma_j^{-1}),\gamma_{j+1},\dots,\gamma_r\big)^{-1}\\ \times\Gamma^{(r)}_{G,\rho}\big(g;(\gamma_j\gamma_1\gamma_j^{-1}),\dots,(\gamma_j\gamma_{j-1}\gamma_j^{-1}),\gamma_j,\gamma_{j+1},\dots,\gamma_r\big).\end{aligned} \tag{3.6}$$

In particular, if γ_j satisfies the condition that $\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1}\in\ker\rho$ for any $1\leqslant i\leqslant r$, then we have

$$\Gamma^{(r)}_{G,\rho}\big(\rho(\gamma_j)g;\underline{\gamma}\big)=\Gamma^{(r)}_{G,\rho}\big(g\rho(\gamma_j);\underline{\gamma}\big)=\Gamma^{(r-1)}_{G,\rho}(g;\check{\underline{\gamma}}_j)^{-1}\Gamma^{(r)}_{G,\rho}(g;\underline{\gamma}), \tag{3.7}$$

where we put $\check{\underline{\gamma}}_j=(\gamma_1,\dots,\gamma_{j-1},\gamma_{j+1},\dots,\gamma_r)$.

Proof. The proof follows immediately from Eq. (3.8) in the following lemma (see, e.g., [13]). The latter follows from the uniqueness of the analytic continuation of the Dirichlet series $\sum_n \mathrm{tr}(A_nB_n)^{-s}$. □

Lemma 3.2. *For a sequence $\mathbf{a}=\{a_n\}_{n\in I}$, whenever the appearing dot-products exist, we have*

$$\prod_{n\in I\amalg J}\!\!\boxed{\bullet}\; a_n=\prod_{n\in I}\!\boxed{\bullet}\; a_n\prod_{n\in J}\!\boxed{\bullet}\; a_n, \tag{3.8}$$

$$\prod_{n\in I}\!\boxed{\bullet}\; a_n^k=\Big(\prod_{n\in I}\!\boxed{\bullet}\; a_n\Big)^k\quad(k>0), \tag{3.9}$$

$$\prod_{n\in I}\!\boxed{\bullet}\;\tau a_n=\exp\left(\sum_{l=1}^{N+1}\frac{(-1)^{l-1}}{l!}(\log\tau)^l\operatorname*{Res}_{s=0}\zeta_{\mathbf{a}}(s)s^{l-2}\right)\prod_{n\in I}\!\boxed{\bullet}\; a_n\quad(\tau>0), \tag{3.10}$$

$$\prod_{n\in I}\!\boxed{\bullet}\;\overline{a_n}=\overline{\prod_{n\in I}\!\boxed{\bullet}\; a_n}. \tag{3.11}$$

Here N in (3.10) *denotes the order of the pole of the zeta function $\zeta_{\mathbf{a}}(s)$ at $s=0$.*

Using the translation laws of $\Gamma^{(r)}_{G,\rho}(g;\underline{\gamma})$, we have the following periodicity of the sine functions, which is a generalization of the formula established in Theorem 2.1 in [17].

Corollary 3.3. *Let $g \in GL(V)$. Then we have*

$$\operatorname{tr}(g)S^{(1)}_{G,\rho}(\rho(\gamma)g;\gamma)=\operatorname{tr}(g^{-1})S^{(1)}_{G,\rho}(g;\gamma). \tag{3.12}$$

Moreover, when γ_a for $1 \leqslant a \leqslant r$ satisfies the condition $\gamma_a\gamma_i\gamma_a^{-1}\gamma_i^{-1} \in \ker\rho$ ($1 \leqslant i \leqslant r$), we have

$$S^{(r,L)}_{G,\rho}(\rho(\gamma_a)g;\underline{\gamma})=S^{(r,L)}_{G,\rho}(g\rho(\gamma_a);\underline{\gamma})=S^{(r-1,L)}_{G,\rho}(g;\check{\underline{\gamma}}_a)^{-1}S^{(r,L)}_{G,\rho}(g;\underline{\gamma}), \tag{3.13}$$

$$S^{(r,R)}_{G,\rho}(\rho(\gamma_a)g;\underline{\gamma})=S^{(r,R)}_{G,\rho}(g\rho(\gamma_a);\underline{\gamma})=S^{(r-1,R)}_{G,\rho}(g;\check{\underline{\gamma}}_a)^{-1}S^{(r,R)}_{G,\rho}(g;\underline{\gamma}), \tag{3.14}$$

and

$$S^{(r)}_{G,\rho}(\rho(\gamma_a)g;\underline{\gamma})=S^{(r)}_{G,\rho}(g\rho(\gamma_a);\underline{\gamma})=S^{(r-1)}_{G,\rho}(g;\check{\underline{\gamma}}_a)^{-1}S^{(r)}_{G,\rho}(g;\underline{\gamma}). \tag{3.15}$$

Proof. Assume that $\operatorname{tr}(g^{-1})\neq 0$.

$$\begin{aligned}\frac{S^{(1)}_{G,\rho}(g;\gamma)}{S^{(1)}_{G,\rho}(\rho(\gamma)g;\gamma)}&=\frac{\Gamma^{(1)}_{G,\rho}(\rho(\gamma)g;\gamma)\Gamma^{(1)}_{G,\rho}(\rho(\gamma)g^{-1};\gamma)}{\Gamma^{(1)}_{G,\rho}(g;\gamma)\Gamma^{(1)}_{G,\rho}(g^{-1}\rho(\gamma);\gamma)}\\&=\frac{\operatorname{tr}(g)\Gamma^{(1)}_{G,\rho}(g;\gamma)\Gamma^{(1)}_{G,\rho}(\rho(\gamma)g^{-1};\gamma)}{\operatorname{tr}(g^{-1})\Gamma^{(1)}_{G,\rho}(\rho(\gamma)g^{-1};\gamma)\Gamma^{(1)}_{G,\rho}(g;\gamma)}=\frac{\operatorname{tr}(g)}{\operatorname{tr}(g^{-1})}.\end{aligned}$$

Hence the (3.12) follows. Next, suppose that γ_a satisfies the required condition. Then, by the translation law of the gamma function (3.7) (or Eq. (3.8)), it follows that

$$\begin{aligned}&S^{(r,R)}_{G,\rho}(g\rho(\gamma_a);\underline{\gamma})\\&\quad=\Gamma^{(r)}_{G,\rho}(g\rho(\gamma_a);\underline{\gamma})^{-1}\Gamma^{(r)}_{G,\rho}(\rho(\gamma_a)^{-1}g^{-1}\rho(\underline{\gamma});\underline{\gamma})^{(-1)^r}\\&\quad=\left\{\Gamma^{(r-1)}_{G,\rho}(g;\check{\underline{\gamma}}_a)^{-1}\Gamma^{(r)}_{G,\rho}(g;\underline{\gamma})\right\}^{-1}\\&\qquad\times\left\{\Gamma^{(r-1)}_{G,\rho}(g^{-1}\rho(\check{\underline{\gamma}}_a);\check{\underline{\gamma}}_a)\Gamma^{(r)}_{G,\rho}(g^{-1}\rho(\underline{\gamma});\underline{\gamma})\right\}^{(-1)^r}\\&\quad=\left\{\Gamma^{(r-1)}_{G,\rho}(g;\check{\underline{\gamma}}_a)^{-1}\Gamma^{(r-1)}_{G,\rho}(g^{-1}\rho(\check{\underline{\gamma}}_a);\check{\underline{\gamma}}_a)^{(-1)^{r-1}}\right\}^{-1}\\&\qquad\times\left\{\Gamma^{(r)}_{G,\rho}(g;\underline{\gamma})^{-1}\Gamma^{(r)}_{G,\rho}(g^{-1}\rho(\underline{\gamma});\underline{\gamma})^{(-1)^r}\right\}\\&\quad=S^{(r-1,R)}_{G,\rho}(g;\check{\underline{\gamma}}_a)^{-1}S^{(r,R)}_{G,\rho}(g;\underline{\gamma}),\end{aligned}$$

because $\rho(\gamma_a)\rho(\gamma_i)=\rho(\gamma_i)\rho(\gamma_a)$. Here, for the second equality, we have calculated as

$$\begin{aligned}&\Gamma^{(r)}_{G,\rho}(\rho(\gamma_a)^{-1}g^{-1}\rho(\underline{\gamma});\underline{\gamma})^{-1}\\&\quad=\boxed{\bullet}_{\underline{\mathbf{n}}\geqslant\underline{0}}\operatorname{tr}(\rho(\underline{\gamma}^{\underline{\mathbf{n}}})\rho(\gamma_a)^{-1}g^{-1}\rho(\underline{\gamma}))\end{aligned}$$

$$
\begin{aligned}
&= \mathop{\boxdot}_{\ell\geqslant 0,\,\underline{\mathbf{m}}\geqslant \underline{0}} \operatorname{tr}\bigl(\rho(\gamma_a)^{\ell}\rho\bigl(\check{\underline{\gamma}}_a^{\underline{\mathbf{m}}}\bigr)\rho(\gamma_a)^{-1}g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr)\rho(\gamma_a)\bigr)\\
&= \mathop{\boxdot}_{\ell\geqslant 0,\,\underline{\mathbf{m}}\geqslant \underline{0}} \operatorname{tr}\bigl(\rho(\gamma_a)^{\ell}\rho\bigl(\check{\underline{\gamma}}_a^{\underline{\mathbf{m}}}\bigr)g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr)\bigr)\\
&= \mathop{\boxdot}_{\underline{\mathbf{m}}\geqslant \underline{0}} \operatorname{tr}\bigl(\rho\bigl(\check{\underline{\gamma}}_a^{\underline{\mathbf{m}}}\bigr)g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr)\bigr) \mathop{\boxdot}_{\ell\geqslant 1,\,\underline{\mathbf{m}}\geqslant \underline{0}} \operatorname{tr}\bigl(\rho(\gamma_a)^{\ell}\rho\bigl(\check{\underline{\gamma}}_a^{\underline{\mathbf{m}}}\bigr)g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr)\bigr)\\
&= \Gamma_{G,\rho}^{(r-1)}\bigl(g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr);\check{\underline{\gamma}}_a\bigr)^{-1} \mathop{\boxdot}_{\ell\geqslant 0,\,\underline{\mathbf{m}}\geqslant \underline{0}} \operatorname{tr}\bigl(\rho(\gamma_a)^{\ell}\rho\bigl(\check{\underline{\gamma}}_a^{\underline{\mathbf{m}}}\bigr)g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr)\rho(\gamma_a)\bigr)\\
&= \Gamma_{G,\rho}^{(r-1)}\bigl(g^{-1}\rho\bigl(\check{\underline{\gamma}}_a\bigr);\check{\underline{\gamma}}_a\bigr)^{-1}\Gamma_{G,\rho}^{(r)}\bigl(g^{-1}\rho(\underline{\gamma});\underline{\gamma}\bigr)^{-1}.
\end{aligned}
$$

Hence the periodicity of the right sine function $S_{G,\rho}^{(r,R)}(g;\underline{\gamma})$ follows. The proof for the $S_{G,\rho}^{(r,R)}(g;\underline{\gamma})$ and $S_{G,\rho}^{(r)}(g;\underline{\gamma})$ can be done similarly. $\square$

Example 3.1. Let $G=SL_2(\mathbb{R})$ and let ρ be the standard representation on $\mathbb{C}^2$. Set $\gamma_1=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $\gamma_2=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in G$, and $g=\left(\begin{smallmatrix}x&y\\z&w\end{smallmatrix}\right)\in \mathrm{Mat}_2(\mathbb{C})(\cong \operatorname{End}\mathbb{C}^2)$. Since

$$
\gamma_1^n\gamma_2^m g=\begin{pmatrix} x+mz & y+mw\\ nx+(nm+1)z & ny+(nm+1)w\end{pmatrix},
$$

the attached zeta function $\zeta_G(s,g;(\gamma_1,\gamma_2))$ is given by

$$
\begin{aligned}
\zeta_G\bigl(s,g;(\gamma_1,\gamma_2)\bigr) &:= \sum_{n,m=0}^{\infty}\bigl\{\operatorname{tr}(\gamma_1^n\gamma_2^m g)\bigr\}^{-s}\\
&= \sum_{n,m=0}^{\infty}(x+w+ny+mz+nmw)^{-s}.
\end{aligned}
$$

Clearly, the series converges absolutely for $\operatorname{Re}s>2$. We see that $\zeta_G(s,g;(\gamma_1,\gamma_2))$ can be meromorphically continued to a region containing $s=0$. First, note that

$$
x+w+ny+mz+nmw=w\left[\left(m+\frac{y}{w}\right)\left(n+\frac{z}{w}\right)+\frac{\det g+w^2}{w^2}\right].
$$

For simplicity, we assume that

$$
(3.16)\qquad y,z>w>0,\qquad \left|\frac{\det g+w^2}{yz}\right|<1\quad\text{and}\quad\left|\frac{\det g+w^2}{w^2}\right|<1.
$$

Then we can calculate as

$$
\begin{aligned}
&\zeta_G\bigl(s,g;(\gamma_1,\gamma_2)\bigr)\\
&= w^{-s}\sum_{n,m=0}^{\infty}\left(m+\frac{y}{w}\right)^{-s}\left(n+\frac{z}{w}\right)^{-s}\left[1+\frac{\det g+w^2}{w^2(m+y/w)(n+z/w)}\right]^{-s}
\end{aligned}
$$

$$= w^{-s}\sum_{n,m=0}^{\infty}\left(m+\frac{y}{w}\right)^{-s}\left(n+\frac{z}{w}\right)^{-s}\sum_{\ell=0}^{\infty}\binom{-s}{\ell}\frac{(\det g + w^2)^{\ell}}{w^{2\ell}(m+y/w)^{\ell}(n+z/w)^{\ell}}$$

$$= w^{-s}\sum_{\ell=0}^{\infty}\binom{-s}{\ell}(w^{-2}\det g+1)^{\ell}\zeta(s+\ell, y/w)\zeta(s+\ell, z/w),$$

where the change of the summations we made in the third equality above can be justified easily. Since $|\zeta(\ell, y/w)| < 1$, $|\zeta(\ell, z/w)| < 1$ for sufficiently large ℓ, and the Hurwitz zeta function $\zeta(s,t)$ is meromorphic in $s \in \mathbb{C}$, we see that the last expression above gives the meromorphic continuation of the zeta function. Especially, we have the Laurent expansion of $\zeta_G(s, g; (\gamma_1, \gamma_2))$ at $s = 0$ as

$$\begin{aligned}\zeta_G\bigl(s, g; (\gamma_1, \gamma_2)\bigr) &= w^{-s}\bigl[\zeta(s, y/w)\zeta(s, z/w)\\ &\quad - (w^{-2}\det g + 1)s\zeta(s+1, y/w)\zeta(s+1, z/w)\bigr] + \mathrm{O}(s^2).\end{aligned}$$

Since $\zeta(s,t) = \frac{1}{2} - t + \{\log\Gamma(t) - (\log 2\pi)/2\}s + c_1(t)s^2 + \mathrm{O}(s^3)$ and $\frac{\partial}{\partial t}\zeta(s,t) = -s\zeta(s+1,t)$, we have $\zeta(s+1,t) = \frac{1}{s} - \frac{\Gamma'(t)}{\Gamma(t)} + \gamma_1(t)s + \mathrm{O}(s^2)$ around $s = 0$, where $\gamma_1(t) = -2c_1'(t)$.

From the observation above, the gamma function can be defined:

$$\text{(3.17)}\qquad \begin{aligned}\Gamma^{(2)}_{SL_2}(g; \gamma_1, \gamma_2) &:= \Gamma^{(2)}_{G,\rho}\bigl(g; (\gamma_1, \gamma_2)\bigr)\\ &= \left\{\prod_{n,m=0}^{\infty}\!\!\boxed{\bullet}\ (x + w + ny + mz + nmw)\right\}^{-1}.\end{aligned}$$

Then the translation laws of the gamma function $\Gamma^{(2)}_{SL_2}(g\gamma_1; \gamma_1, \gamma_2)$ can be described as

$$\begin{aligned}\Gamma^{(2)}_{SL_2}(g\gamma_1; \gamma_1, \gamma_2) &= \Gamma^{(1)}_{G,\rho}(g; \gamma_2)^{-1}\Gamma^{(2)}_{SL_2}(g; \gamma_1, \gamma_2)\\ &= \prod_{n=0}^{\infty}(x + w + nz)\,\Gamma^{(2)}_{SL_2}(g; \gamma_1, \gamma_2),\\ \Gamma^{(2)}_{SL_2}(\gamma_2 g; \gamma_1, \gamma_2) &= \Gamma^{(1)}_{G,\rho}(g; \gamma_1)^{-1}\Gamma^{(2)}_{SL_2}(g; \gamma_1, \gamma_2)\\ &= \prod_{m=0}^{\infty}(x + w + my)\,\Gamma^{(2)}_{SL_2}(g; \gamma_1, \gamma_2),\end{aligned}$$

that is, we obtain

$$\text{(3.18)}\qquad \begin{aligned}&\Gamma^{(2)}_{SL_2}\left(\begin{pmatrix} x+y & y \\ z+w & w\end{pmatrix}; \gamma_1, \gamma_2\right)\\ &\quad = \sqrt{2\pi}\, y^{\frac{1}{2}-\frac{x+w}{y}}\,\Gamma\left(\frac{x+w}{z}\right)^{-1}\Gamma^{(2)}_{SL_2}\left(\begin{pmatrix} x & y \\ z & w\end{pmatrix}; \gamma_1, \gamma_2\right),\end{aligned}$$

$$(3.19)\quad \Gamma^{(2)}_{SL_2}\left(\begin{pmatrix} x+z & y+w \\ z & w \end{pmatrix}; \gamma_1, \gamma_2\right) = \sqrt{2\pi}\, z^{\frac{1}{2}-\frac{x+w}{z}}\, \Gamma\left(\frac{x+w}{y}\right)^{-1} \Gamma^{(2)}_{SL_2}\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \gamma_1, \gamma_2\right),$$

where we have used (3.10) together with the Lerch formula; $\prod_{n=0}^{\infty} y(t+n) = y^{\zeta(0,t)} \prod_{n=0}^{\infty}(t+n) = \sqrt{2\pi}\, y^{\frac{1}{2}-t}/\Gamma(t)$.

Example 3.2. Let $G = GL_2(\mathbb{R})$ and ρ be the standard representation of G. Put $\gamma_1 = \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $q > 1$ and $\gamma_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For $g = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{C})$, we have $\mathrm{tr}(\gamma_1^m \gamma_2^n g) = q^m x + w + nzq^m$. The zeta function $\zeta_G(s, g; \gamma_1, \gamma_2) := \sum_{m,n=0}^{\infty}(q^m x + w + nzq^m)^{-s}$ can be meromorphically extended to $s \in \mathbb{C}$. For simplicity, we assume that $|w| < \min(x, 1)$ and $x > z \geqslant 1$. Then, in fact, since $\zeta_G(s, g; \gamma_1, \gamma_2) = \sum_{m,n=0}^{\infty}(x+nz)^{-s} q^{-ms}(1+q^{-m}\frac{w}{x+nz})^{-s}$, by the calculation similar to the examples above (or, see [32] for details), for $\mathrm{Re}\, s > 1$, we obtain

$$(3.20)\quad \zeta_G(s, g; \gamma_1, \gamma_2) = \sum_{\ell=0}^{\infty} \binom{-s}{\ell} \frac{\zeta(s+\ell, \frac{x}{z}) w^{\ell} z^{-s-\ell}}{1 - q^{-(s+\ell)}}.$$

Since $\zeta(s, x/z)$, $s\zeta(s+1, x/z)$, and $\zeta(s+\ell, x/z)$ ($\ell \geqslant 2$) are holomorphic at $s = 0$, the meromorphy of $\zeta_G(s, g; \gamma_1, \gamma_2)$ follows from the fact $|\zeta(\ell, x/z)| < 1$ for sufficiently large ℓ. Hence, we have the following (non-constant) gamma function attached to this data.

$$(3.21)\quad \Gamma^{(2)}_{GL_2}(g; \gamma_1, \gamma_2) := \Gamma^{(2)}_{G,\rho}(g; \gamma_1, \gamma_2) = \left\{ \boxed{\prod_{n,m=0}^{\infty}} \left(q^m x + w + nzq^m\right) \right\}^{-1}.$$

Then by (3.4) and (3.5), it can be shown that the attached gamma function $\Gamma^{(2)}_{GL_2}(g; \gamma_1, \gamma_2)$ satisfies

$$\begin{aligned} \Gamma^{(2)}_{GL_2}(g\gamma_1; \gamma_1, \gamma_2) &= \Gamma^{(1)}_{GL_2}(g, \gamma_2)^{-1} \Gamma^{(2)}(g; \gamma_1, \gamma_2) \\ &= \prod_{n=0}^{\infty}(x + w + nz)\, \Gamma^{(2)}_{GL_2}(g; \gamma_1, \gamma_2), \\ \Gamma^{(2)}_{GL_2}(\gamma_2 g; \gamma_1, \gamma_2) &= \Gamma^{(1)}_{GL_2}(g, \gamma_1)^{-1} \Gamma^{(2)}(g; \gamma_1, \gamma_2) \\ &= \boxed{\prod_{m=0}^{\infty}}(q^m x + w)\, \Gamma^{(2)}_{GL_2}(g; \gamma_1, \gamma_2), \end{aligned}$$

that is, we have

$$(3.22)\quad \Gamma^{(2)}_{GL_2}\left(\begin{pmatrix} qx & y \\ qz & w \end{pmatrix}; \gamma_1, \gamma_2\right) = \sqrt{2\pi}\, z^{\frac{1}{2}-\frac{x+w}{z}}\, \Gamma\left(\frac{x+w}{z}\right)^{-1} \Gamma^{(2)}_{GL_2}\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \gamma_1, \gamma_2\right),$$

(3.23) $$\Gamma^{(2)}_{GL_2}\left(\begin{pmatrix} x+z & y+w \\ z & w \end{pmatrix}; \gamma_1, \gamma_2\right)$$
$$= q^{-\frac{1}{12}} x^{\frac{1}{2}-\frac{1}{2}\log_q x} G\left(q^{-1}, -\frac{w}{x}\right) \Gamma^{(2)}_{GL_2}\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \gamma_1, \gamma_2\right),$$

where $G(q^{-1}, z) = \prod_{n=0}^{\infty}(1 - q^{-n}z)$. The first translation law is essentially the same as the previous example and the second one can be shown in a similar way to Theorem 2.1. So we leave it to the reader. We may call this gamma function $\Gamma^{(2)}_{GL_2}(g; \gamma_1, \gamma_2)$ a *hybrid* double gamma function because it has both the classical and the quantum characteristics at the same time as we have seen above.

By Proposition 3.1 it is easy to check the following

Lemma 3.4. *Let* $\mathfrak{g}$ *be the Lie algebra of* G. *Suppose that* γ_0 *is written as* $\gamma_0 = e^{\omega_0 X}$ *for some* $X \in \mathfrak{g}$ *and* $\omega_0 \in \mathbb{C}$. *Let* $h \in \mathrm{End}(V)$. *Then*

(3.24)
$$\frac{\Gamma^{(r+1)}_{G,\rho}(h\rho(e^{(t+\omega_0)X}); (e^{\omega_0 X}, \gamma_1, \ldots, \gamma_r))\Gamma^{(r+1)}_{G,\rho}(h\rho(e^{-tX}); (e^{\omega_0 X}, \gamma_1^{-1}, \ldots, \gamma_r^{-1}))}{\Gamma^{(r+1)}_{G,\rho}(h\rho(e^{tX}); (e^{\omega_0 X}, \gamma_1, \ldots, \gamma_r))\Gamma^{(r+1)}_{G,\rho}(h\rho(e^{-(t-\omega_0)X}); (e^{\omega_0 X}, \gamma_1^{-1}, \ldots, \gamma_r^{-1}))}$$
$$= \frac{\Gamma^{(r)}_{G,\rho}(h\rho(e^{-tX}); (\gamma_1^{-1}, \ldots, \gamma_r^{-1}))}{\Gamma^{(r)}_{G,\rho}(h\rho(e^{tX}); (\gamma_1, \ldots, \gamma_r))}.$$

Let us consider the situation that $h \in \mathrm{End}(V)$ satisfies

(3.25) $$\mathrm{tr}\big(\rho(g)^{-1}h\big) = -\,\mathrm{tr}\big(\rho(g)h\big) \quad (\forall g \in G).$$

Further, assume that $\gamma_i\gamma_j\gamma_i^{-1}\gamma_j \in \ker\rho$ for $i, j = 0, 1, \ldots, r$. Then

$$\begin{aligned}
&\zeta^{(r)}_{G,\rho}\big(s, h\rho(e^{-tX}); (\gamma_1^{-1}, \ldots, \gamma_r^{-1})\big) \\
&\quad = \sum_{n_1,\ldots,n_r \geqslant 0} \big\{\mathrm{tr}\big(h\rho(e^{-tX})\rho(\gamma_1^{-n_1}\cdots\gamma_r^{-n_r})\big)\big\}^{-s} \\
&\quad = \sum_{n_1,\ldots,n_r \geqslant 0} \big\{-\mathrm{tr}\big(h\rho(e^{tX})\rho(\gamma_1^{n_1}\cdots\gamma_r^{n_r})\big)\big\}^{-s}.
\end{aligned}$$

Hence, if all the "arguments of $\rho(e^{tX}\gamma_1^{n_1}\cdots\gamma_r^{n_r})$" are positive (resp. negative) we have

$$\zeta^{(r)}_{G,\rho}\big(s, h\rho(e^{-tX}); (\gamma_1^{-1}, \ldots, \gamma_r^{-1})\big) = e^{\pm\pi i s}\zeta^{(r)}_{G,\rho}\big(s, h\rho(e^{tX}); (\gamma_1, \ldots, \gamma_r)\big),$$

respectively, according to our fixed choice of the log-argument. Therefore, since we are assuming the zeta function $\zeta^{(r)}_{G,\rho}(s, h\rho(e^{tX}); (\gamma_1, \ldots, \gamma_r))$ is meromorphic at $s = 0$, it follows that

$$
\begin{aligned}
&\log\Gamma^{(r)}_{G,\rho}\big(h\rho(e^{-tX});(\gamma_1^{-1},\dots,\gamma_r^{-1})\big)\\
&\quad=-\frac{1}{2}\pi^2\operatorname*{Res}_{s=0}\zeta^{(r)}_{G,\rho}\big(s,h\rho(e^{tX});(\gamma_1,\dots,\gamma_r)\big)\\
&\qquad\pm\pi i\operatorname*{Res}_{s=0}\frac{\zeta^{(r)}_{G,\rho}(s,h\rho(e^{tX});(\gamma_1,\dots,\gamma_r))}{s}\\
&\qquad+\log\Gamma^{(r)}_{G,\rho}\big(h\rho(e^{tX});(\gamma_1,\dots,\gamma_r)\big).
\end{aligned}
$$

This shows that the right-hand side of (3.24) is equal to

$$
\begin{aligned}
\exp\Bigg\{&-\frac{1}{2}\pi^2\operatorname*{Res}_{s=0}\zeta^{(r)}_{G,\rho}\big(s,h\rho(e^{tX});(\gamma_1,\dots,\gamma_r)\big)\\
&\pm\pi i\operatorname*{Res}_{s=0}\frac{\zeta^{(r)}_{G,\rho}(s,h\rho(e^{tX});(\gamma_1,\dots,\gamma_r))}{s}\Bigg\}.
\end{aligned}
$$

Example 3.3. Let $\gamma_i=e^{\omega_i X}$, $X=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\in\mathfrak{sl}_2(\mathbb{C})$ and suppose that $\operatorname{Im}\omega_i>0$ for all i. Consider the standard representation of $G=SL_2(\mathbb{C})$. Take $h=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then h satisfies the condition (3.25) for G. It is immediate to see that the zeta function $\zeta^{(r)}_{G,\rho}(s,h\rho(e^{tX});(\gamma_1,\dots,\gamma_r))$ in this case is the multiple Hurwitz zeta function $\zeta_r(s,t,\underline{\omega})=\sum_{n_1,\dots,n_r=0}^{\infty}(t+n_1\omega_1+\cdots+n_r\omega_r)^{-s}$ and hence the corresponding gamma function is the Barnes gamma function $\Gamma_r(t,\underline{\omega})=\prod_{n_1,\dots,n_r=0}(t+n_1\omega_1+\cdots+n_r\omega_r)^{-1}$. Since $\zeta_r(s,t,\underline{\omega})$ is holomorphic at $s=0$, from the lemma above we obtain

$$
\frac{\Gamma_{r+1}(t+\omega_0,(\omega_0,\underline{\omega}))\Gamma_{r+1}(-t,(\omega_0,-\underline{\omega}))}{\Gamma_{r+1}(t,(\omega_0,\underline{\omega}))\Gamma_{r+1}(-t+\omega_0,(\omega_0,-\underline{\omega}))}=e^{\pi i\zeta_r(0,t,\underline{\omega})}. \tag{3.26}
$$

Remark. Comparing the poles of the Barnes gamma function [3], we notice that there exists an entire function $f_{r+1}(t,(\omega_0,\underline{\omega}))$ such that

$$
\begin{aligned}
&\Gamma_{r+1}\big(t,(\omega_0,\underline{\omega})\big)\Gamma_{r+1}\big(-t+\omega_0,(\omega_0,-\underline{\omega})\big)\\
&\quad=e^{-\pi i f_{r+1}(t,(\omega_0,\underline{\omega}))}\prod_{\underline{\mathbf{n}}\geqslant\underline{0}}\big(1-e^{2\pi i(t+\underline{\mathbf{n}}\cdot\underline{\omega})}\big)^{-1}.
\end{aligned}
$$

From (3.26), it is easy to see that

$$
f_{r+1}\big(t+\omega_0,(\omega_0,\underline{\omega})\big)-f_{r+1}\big(t,(\omega_0,\underline{\omega})\big)=\zeta_r(0,t,\underline{\omega}).
$$

Thus, it is quite natural to expect that ζ_{r+1} gives f_{r+1}. It is actually true. Indeed, recently, it was proved in [9] that $f_{r+1}(t,(\omega_0,\underline{\omega}))=\zeta_{r+1}(0,t,(\omega_0,\underline{\omega}))$ using the Raabe type formula discussed in the subsequent section. It generalizes the relation between Barnes' double gamma function and the modular function appearing in the Kronecker second limit formula in [30] (see [9]) when $r=1$.

It has been discussed quite recently in [4,6,9] that

$$\int_0^1 \zeta(s,x)\,dx = 0 \tag{4.1}$$

when $\operatorname{Re} s < 1$. This fact follows from a simple integral equation described in the following lemma, which is derived from the ladder structure of the zeta function. See Example 4.1 below.

We regard the integral above as a period integral because the interval $(0,1]$ is the fundamental domain for the action of the additive group $\mathbb{Z}$ on $\mathbb{R}$. (It is also considered as the constant term of the Fourier expansion of the integrand.)

Lemma 4.1. *Let $\mathfrak{g}$ be a Lie algebra of the Lie group G. Consider the zeta function $\zeta_{G,\rho}(s,g,\underline{\gamma}) = \sum_{\underline{\mathbf{n}}\geqslant\underline{0},\,\underline{\mathbf{n}}\in\mathbb{Z}^r_{\geqslant 0}} \operatorname{tr}(\rho(\underline{\gamma}^{\underline{\mathbf{n}}})g)^{-s}$ attached to a representation (ρ, V) of G, where $\underline{\gamma} = (\gamma_1,\dots,\gamma_r)$. Write $\gamma_1 = e^{X_1}$ for $X_1 \in \mathfrak{g}$. Suppose that there exists a function $F_s(t,X)$ on $\mathbb{R}\times\operatorname{End}(V)$ satisfying the following conditions:*

(1) $\frac{\partial}{\partial t}F_s(t,g) = \operatorname{tr}(e^{tX}g)^{-s}$,
(2) *there is a positive real number a such that $\lim_{t\to\infty} F_s(t,\rho(\underline{\gamma}^{\underline{\mathbf{n}}})g) = 0$ when $\operatorname{Re} s > a$ uniformly for $g \in U$ for any compact subset U of $\operatorname{End}(V)$ and $\underline{\mathbf{n}} \in \mathbb{Z}^r_{\geqslant 0}$.*

Then we have for $x\in\mathbb{R}$ and $\operatorname{Re} s > a$

$$\int_x^{x+1} \zeta^{(r)}_{G,\rho}\big(s, h\rho(e^{tX_1}), \underline{\gamma}\big)\,dt = -\sum_{n_2,\dots,n_r=0}^{\infty} F_s\big(x, \rho(\gamma_2^{n_2}\cdots\gamma_r^{n_r})h\big).$$

Proof. By the condition, we may change the order of the integration and summation:

$$\begin{aligned}
&\int_x^{x+1} \zeta^{(r)}_{G,\rho}\big(s, h\rho(e^{tX_1}), \underline{\gamma}\big)\,dt \\
&= \int_x^{x+1} \sum_{n_1,n_2,\dots,n_r=0}^{\infty} \operatorname{tr}\big(\rho\big(e^{(t+n_1)X_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r}\big)h\big)^{-s}\,dt \\
&= \sum_{n_2,\dots,n_r=0}^{\infty}\sum_{n_1=0}^{\infty}\big[F_s\big(t+n_1, \rho(\gamma_2^{n_2}\cdots\gamma_r^{n_r})h\big)\big]_x^{x+1} \\
&= -\sum_{n_2,\dots,n_r=0}^{\infty} F_s\big(x, \rho(\gamma_2^{n_2}\cdots\gamma_r^{n_r})h\big). \qquad \square
\end{aligned}$$

Example 4.1. Let $\gamma_j = \begin{pmatrix} 1 & \omega_j \\ 0 & 1 \end{pmatrix} = \exp X_j \in SL_2(\mathbb{C})$, where $X_j := \omega_j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, as before, the zeta function $\zeta^{(r)}_{G,\rho}(s, g; \underline{\gamma})$ for the standard representation ρ of $SL_2(\mathbb{C})$ is given by $\zeta^{(r)}_{G,\rho}(s, g; \underline{\gamma}) = \zeta_r(s, \operatorname{tr} g, z\underline{\omega})$ when $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \mathbb{C}^r$. Hence, for $\operatorname{Re} s > r$, we have from the lemma above that

$$\begin{aligned}
&\int_{x_1}^{x_1+1} \zeta^{(r)}_{G,\rho}(s, ge^{t_1 X_1}; \underline{\gamma})\, dt_1 \\
&= \sum_{n_2,\ldots,n_r=0}^{\infty} \frac{1}{(s-1)z\omega_1} \big(\operatorname{tr} g + z\{x_1\omega_1 + n_2\omega_2 + \cdots + n_r\omega_r\}\big)^{-s+1}.
\end{aligned}$$

The change of the order of the integration and summation above can be easily justified. Repeating this process, we find that

$$\begin{aligned}
&\int_{x_1}^{x_1+1} \cdots \int_{x_r}^{x_r+1} \zeta^{(r)}_{G,\rho}(s, ge^{t_1 X_1} \cdots e^{t_r X_r}; \underline{\gamma})\, dt_1 \cdots dt_r \\
&= \frac{1}{z^r \prod_{k=1}^r (s-k) \prod_{k=1}^r \omega_k} \big(\operatorname{tr} g + z\{x_1\omega_1 + \cdots + x_r\omega_r\}\big)^{-s+r}
\end{aligned} \tag{4.2}$$

for $\operatorname{Re} s > r$. Obviously, the right hand side, as well as the left hand side, can be meromorphically continued to the whole plane $\mathbb{C}$. Now we assume that the both sides are meromorphically extended to $\operatorname{Re} s < r$. Then, using the Cauchy integral theorem, we have

$$\begin{aligned}
&\int_{x_1}^{x_1+1} \cdots \int_{x_r}^{x_r+1} \operatorname*{Res}_{s=0} \frac{\zeta^{(r)}_{G,\rho}(s, ge^{t_1 X_1} \cdots e^{t_r X_r}; \underline{\gamma})}{s^2}\, dt_1 \cdots dt_r \\
&= \frac{1}{z^r \prod_{k=1}^r \omega_k} \cdot \operatorname*{Res}_{s=0} \frac{1}{\prod_{k=1}^r (s-k)} \cdot \frac{1}{s^2} \big(\operatorname{tr} g + z\{x_1\omega_1 + \cdots + x_r\omega_r\}\big)^{-s+r} \\
&= \frac{(-1)^r}{z^r r! \prod_{k=1}^r \omega_k} \big(\operatorname{tr} g + z\{x_1\omega_1 + \cdots + x_r\omega_r\}\big)^{r} \\
&\quad \times \big\{-\log\big(\operatorname{tr} g + z\{x_1\omega_1 + \cdots + x_r\omega_r\}\big) + H_r\big\},
\end{aligned}$$

where we put $H_r = 1 + 1/2 + \cdots + 1/r$.

By the definition of the gamma function associated with this zeta function, it follows that

$$\begin{aligned}
&\int_{x_1}^{x_1+1} \cdots \int_{x_r}^{x_r+1} \log \Gamma^{(r)}_{G,\rho}\big(ge^{t_1 X_1} \cdots e^{t_r X_r}; \underline{\gamma}\big)\, dt_1 \cdots dt_r \\
&= \int_{x_1}^{x_1+1} \cdots \int_{x_r}^{x_r+1} \log \Gamma_r\big(\operatorname{tr} g + z\{t_1\omega_1 + \cdots + t_r\omega_r\}\big)\, dt_1 \cdots dt_r
\end{aligned} \tag{4.3}$$

$$= \frac{(-1)^{r-1}}{z^r r! \prod_{k=1}^r \omega_k} \bigl(\operatorname{tr} g + z\{x_1\omega_1 + \cdots + x_r\omega_r\}\bigr)^r \times \bigl\{\log\bigl(\operatorname{tr} g + z\{x_1\omega_1 + \cdots + x_r\omega_r\}\bigr) - H_r\bigr\}$$

for the multiple gamma function $\Gamma_r(x)$. Thus, if we put $\operatorname{tr} g = x_1 = \cdots = x_r = 0$, we have

$$\int_0^1 \cdots \int_0^1 \log \Gamma_r(t_1\omega_1 + \cdots + t_r\omega_r)\, dt_1 \cdots dt_r = 0. \tag{4.4}$$

If we take $r = 1$, we get

$$\int_0^1 \log \Gamma(t)\, dt = \frac{1}{2} \log 2\pi \tag{4.5}$$

from the Lerch formula $\Gamma_1(x) = \prod_{n=0}^\infty (n+x)^{-1} = \Gamma(x)/\sqrt{2\pi}$. This is known as the Raabe formula. Note that this is equivalent with the Euler 's famous formula $\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2$ in [7]. The discussion made above can be applied to the case of the Shintani (and its slightly generalized version of) zeta functions [29]. Indeed, since the Shintani gamma function is given by a product of the Barnes multiple gamma functions [32] (up to a polynomial factor, and this fact was already clarified in [29]), the corresponding formula (4.3) for the Shintani gamma function is directly obtained from the above one. Putting $\operatorname{tr} g = x_1 = \cdots = x_r = 0$ in the analytically continued one of (4.2), we have $\int_0^1 \cdots \int_0^1 \zeta_r(s, t_1\omega_1 + \cdots + t_r\omega_r, \underline{\omega})\, dt_1 \cdots dt_r = 0$ when $\operatorname{Re} s < r$. In [9] one can find the same formula for the Shintani zeta function.

Example 4.2. Recall the situation in Example 3.1: Let $G = SL_2(\mathbb{R})$ and let ρ be the standard representation on $\mathbb{C}^2$. Set $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\gamma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \exp Y$ and $\gamma_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \exp X$. Hence, for $g = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \operatorname{End} \mathbb{C}^2$, we see that

$$\begin{aligned} \zeta_G\bigl(s, ge^{tY}; (\gamma_1, \gamma_2)\bigr) &= \sum_{n,m=0}^\infty \bigl\{\operatorname{tr}\bigl(e^{(t+n)Y} e^{mX} g\bigr)\bigr\}^{-s} \\ &= \sum_{n,m=0}^\infty \bigl(x + w + ny + mz + nmw + t(y + mw)\bigr)^{-s}. \end{aligned}$$

We consider the following period integral $I(\alpha, \beta)$ for $\alpha, \beta \geqslant 0$ defined by

$$I(\alpha, \beta) := \int_\beta^{\beta+1} \int_\alpha^{\alpha+1} \zeta_G\bigl(s, e^{pX} g e^{tY}; (\gamma_1, \gamma_2)\bigr)\, dt\, dp. \tag{4.6}$$

First, by Lemma 4.1 we note that

$$\int_{\alpha}^{\alpha+1} \zeta_G\big(s, ge^{tY}; (\gamma_1, \gamma_2)\big)\, dt$$

$$= -\frac{1}{1-s} \sum_{m=0}^{\infty} \frac{1}{y+mw} \big(x + w + mz + \alpha(y+mw)\big)^{-s+1}.$$

Therefore we have

$$I(\alpha, \beta) = -\frac{1}{1-s} \sum_{m=0}^{\infty} \int_{\beta}^{\beta+1} \frac{1}{y+(m+p)w}$$

$$\times \big\{x + w + \alpha y + (m+p)(z+\alpha w)\big\}^{-s+1}\, dp$$

$$= -\frac{1}{1-s} \int_{\beta}^{\infty} \frac{1}{y+pw} \big\{x + w + \alpha y + p(z+\alpha w)\big\}^{-s+1}\, dp.$$

- We first consider the case $w = 0$. Then, if $yz \neq 0$ we have

$$I(\alpha, \beta) = \frac{1}{yz} \cdot \frac{1}{(1-s)(2-s)} (x + \alpha y + \beta z)^{-s+2}$$

when $\operatorname{Re} s > 2$. Therefore, by the same procedure in the previous example using the Cauchy integral formula we see that

$$\int_{\beta}^{\beta+1} \int_{\alpha}^{\alpha+1} \log \Gamma_G\left(\begin{pmatrix} x+ty+pz & y \\ z & 0 \end{pmatrix}; (\gamma_1, \gamma_2)\right) dt\, dp$$

$$= -\frac{(x+\alpha y+\beta z)^2}{2yz} \left\{\log(x+\alpha y+\beta z) - \frac{3}{2}\right\}$$

by noting that

$$e^{pX} ge^{tY} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} x+ty+pz & y \\ z & 0 \end{pmatrix}.$$

Hence, in particular, putting $\alpha = \beta = 0$ we have

$$\int_0^1 \int_0^1 \log \Gamma_G\left(\begin{pmatrix} x+ty+pz & y \\ z & 0 \end{pmatrix}; (\gamma_1, \gamma_2)\right) dt\, dp \tag{4.7}$$

$$= -\frac{x^2}{2yz}\left(\log x - \frac{3}{2}\right) \xrightarrow{x \to 0} 0.$$

• Next we consider the case $w \neq 0$. For simplicity, we assume that $\frac{x+w+\alpha y}{z+\alpha w} \geqslant \frac{y}{w} > -\beta$. Then, since

$$0 < \frac{1}{y+pw} = \frac{1}{w}\sum_{\ell=0}^{\infty}\left(\frac{x+w+\alpha y}{z+\alpha w} - \frac{y}{w}\right)^{\ell}\left(p + \frac{x+w+\alpha y}{z+\alpha w}\right)^{-(\ell+1)},$$

we can calculate for $\operatorname{Re} s > 1$ as

$$\begin{aligned} I(\alpha,\beta) &= -\frac{1}{1-s}\cdot\frac{1}{w}(z+\alpha w)^{-s+1}\sum_{\ell=0}^{\infty}\int_{\beta}^{\infty}\left(\frac{x+w+\alpha y}{z+\alpha w} - \frac{y}{w}\right)^{\ell} \\ &\quad\times\left(p+\frac{x+w+\alpha y}{z+\alpha w}\right)^{-s-\ell} dp \\ &= -\frac{1}{1-s}\cdot\frac{1}{w}(z+\alpha w)^{-s+1}\sum_{\ell=0}^{\infty}\left(\frac{x+w+\alpha y}{z+\alpha w} - \frac{y}{w}\right)^{\ell} \\ &\quad\times\frac{1}{s+\ell-1}\left(\beta+\frac{x+w+\alpha y}{z+\alpha w}\right)^{-s-\ell+1} \\ &= \frac{1}{s-1}\cdot\frac{1}{w}\sum_{\ell=0}^{\infty}\frac{1}{s+\ell-1}\left(\frac{\det g + w^2}{w}\right)^{\ell} \\ &\quad\times\left(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w\right)^{-s-\ell+1}. \end{aligned}$$

Notice that, at the origin $s = 0$, we have the Laurent expansions:

$$\frac{1}{(s-1)^2} = 1 + 2s + \mathrm{O}(s^2), \qquad \frac{1}{s(s-1)} = -\frac{1}{s} - 1 - s + \mathrm{O}(s^2),$$
$$\frac{1}{(s+\ell-1)(s-1)} = -\frac{1}{\ell-1}\left\{1 + \left(1 - \frac{1}{\ell-1}\right)s\right\} + \mathrm{O}(s^2) \quad \text{for } \ell \geqslant 2$$

and

$$\begin{aligned} &(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w)^{-s} \\ &\quad = 1 - s\log(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w) \\ &\qquad + \frac{s^2}{2}\left\{\log(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w)\right\}^2 + \mathrm{O}(s^3). \end{aligned}$$

These show that

$$\begin{aligned} &w \times \operatorname*{Res}_{s=0}\frac{I(\alpha,\beta)}{s^2} \\ &\quad = (\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w)\left\{2 - \log(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w)\right\} \\ &\qquad + \left(\frac{\det g + w^2}{w}\right)\Big[-1 + \log(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w) \\ &\qquad - \frac{1}{2}\left\{\log(\operatorname{tr} g + \alpha y + \beta z + \alpha\beta w)\right\}^2\Big] \end{aligned}$$

$$
\begin{aligned}
&+\{\log(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)-1\}\\
&\times\sum_{\ell=2}^{\infty}\frac{1}{\ell-1}\left(\frac{\det g+w^2}{w}\right)^{\ell}(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)^{-\ell+1}\\
&+\sum_{\ell=2}^{\infty}\frac{1}{(\ell-1)^2}\left(\frac{\det g+w^2}{w}\right)^{\ell}(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)^{-\ell+1}.
\end{aligned}
$$

By the definition of the gamma function, in terms of the logarithmic and dilogarithmic functions, we obtain

$$
\begin{aligned}
&\int_{\beta}^{\beta+1}\int_{\alpha}^{\alpha+1}\log\Gamma_G\left(\begin{pmatrix} x+pz+ty+ptw & y+pw\\ z+tw & w\end{pmatrix};(\gamma_1,\gamma_2)\right)dt\,dp\\
&=\frac{1}{w}\Bigg[(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)\{2-\log(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)\}\\
&\quad+\left(\frac{\det g+w^2}{w}\right)\Bigg\{-1+\log(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)\\
&\quad-\frac{1}{2}\{\log(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)\}^2\\
&\quad+\{\log(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)-1\}\operatorname{Li}_1\left(\frac{\det g+w^2}{w(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)}\right)\\
&\quad+\operatorname{Li}_2\left(\frac{\det g+w^2}{w(\operatorname{tr} g+\alpha y+\beta z+\alpha\beta w)}\right)\Bigg\}\Bigg],
\end{aligned}
$$

where

$$
\operatorname{Li}_k(z):=\sum_{\ell=1}^{\infty}\frac{z^{\ell}}{\ell^k}\quad(k=1,2,\ldots).
$$

If we put $\alpha=\beta=0$ then

$$
\begin{aligned}
&\int_{0}^{1}\int_{0}^{1}\log\Gamma_G\left(\begin{pmatrix} x+pz+ty+ptw & y+pw\\ z+tw & w\end{pmatrix};(\gamma_1,\gamma_2)\right)dt\,dp\\
&=\frac{1}{w}\Bigg[(\operatorname{tr} g)\{2-\log(\operatorname{tr} g)\}+\left(\frac{\det g+w^2}{w}\right)\\
&\quad\times\Bigg\{\left(-1+\log(\operatorname{tr} g)-\frac{1}{2}(\log(\operatorname{tr} g))^2\right)\\
&\quad+(\log(\operatorname{tr} g)-1)\operatorname{Li}_1\left(\frac{\det g+w^2}{w\operatorname{tr} g}\right)+\operatorname{Li}_2\left(\frac{\det g+w^2}{w\operatorname{tr} g}\right)\Bigg\}\Bigg].
\end{aligned}
$$

Example 4.3. It is easy see from the discussion in Example 3.2 that the q-Hurwitz zeta function $\zeta_q(s,x)=\sum_{n=0}^{\infty}[n+x]_q^{-s}$, where $[x]_q=\frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}$, is regarded as the one for the group representation. (See [32] for a generalized version, i.e., a q-analogue of the Shintani zeta function.) Since

$$\frac{\partial}{\partial x}[n+x]_q^{-s+1}=(-s+1)\frac{\log q^{1/2}}{q^{1/2}-q^{-1/2}}(q^{(n+x)/2}+q^{-(n+x)/2})[n+x]_q^{-s}$$

by the same procedure above, it is not hard to see the relation

$$\begin{aligned}(4.8)\qquad \log q^{1/2}\int_x^{x+1}&\left\{(s-2)\zeta_q(s-2,t)+\frac{4(s-1)}{(q^{1/2}-q^{-1/2})^2}\zeta_q(s,t)\right\}dt\\ &=\frac{q^{x/2}+q^{-x/2}}{q^{1/2}-q^{-1/2}}[x]_q^{-s+1}\end{aligned}$$

when $\operatorname{Re}s>2$. Now recall the formula in [12] (in [31], the formula was already obtained but the convention of q-analogue is different) which gives the analytic continuation of $\zeta_q(s,t)$ to the entire plane $\mathbb{C}$:

$$(4.9)\qquad \zeta_q(s,t)=\left(q^{1/2}-q^{-1/2}\right)^s\sum_{k=0}^{\infty}\binom{s+k-1}{k}\frac{q^{(k+s/2)(1-t)}}{q^{k+s/2}-1}.$$

To describe the formula for the q-gamma function $\Gamma_q(x)$ we now introduce the following function with two parameters $a\in\mathbb{C}$ and $q>1$.

$$\mathcal{O}_{a,q}(x)=\begin{cases}\prod_{n=0}^{\infty}\left\{(1-q^{-(n+x)})e^{\sum_{j=1}^{[\operatorname{Re}a]}\frac{q^{-j(n+x)}}{j}}\right\}^{q^{(n+x)a}} & \text{if } \operatorname{Re}a\geqslant 1,\\ \prod_{n=0}^{\infty}(1-q^{-(n+x)})^{q^{(n+x)a}} & \text{if } \operatorname{Re}a<1,\end{cases}$$

where $[b]$ denotes the integer part of $b\in\mathbb{R}$. One can find that $\mathcal{O}_{a,q}(x)$ has an obvious functional equation for the translation $x\to x+1$. Note also that the q-gamma function is given by $\Gamma_q(x)=\mathcal{O}_q(0)\mathcal{O}_q(x)^{-1}(q^{1/2}-q^{-1/2})^{1-x}q^{x(x-1)/4}$, where $\mathcal{O}_q(x)=\mathcal{O}_{0,q}(x)=G(q^{-1},q^{-x})$ (see [12] for the definition of $\Gamma_q(x)$ which is slightly different from the one in [1]). Moreover, we have

$$(4.10)\qquad \log\mathcal{O}_{a,q}(x)=-\sum_{j=\min\{1,[\operatorname{Re}a]+1\}}^{\infty}\frac{1}{j}\cdot\frac{q^{(j-a)(1-x)}}{q^{j-a}-1}.$$

In fact, for instance, when $\operatorname{Re}a\geqslant 1$, we observe that

$$\begin{aligned}\log\mathcal{O}_{a,q}(x)&=\sum_{n=0}^{\infty}q^{(n+x)a}\left\{\log\left(1-q^{-(n+x)}\right)+\sum_{j=1}^{[\operatorname{Re}a]}\frac{q^{-j(n+x)}}{j}\right\}\\ &=-\sum_{n=0}^{\infty}q^{(n+x)a}\sum_{j=[\operatorname{Re}a]+1}^{\infty}\frac{q^{-j(n+x)}}{j}\end{aligned}$$

$$= - \sum_{j=[\mathrm{Re}\,a]+1}^{\infty} \frac{1}{j} q^{-x(j-a)} \sum_{n=0}^{\infty} q^{-n(j-a)}$$

$$= - \sum_{j=[\mathrm{Re}\,a]+1}^{\infty} \frac{1}{j} q^{-x(j-a)} \frac{1}{1-q^{-(j-a)}},$$

whence the formula (4.10) follows.

Using Eq. (4.9), we have the analytic continuation for the both sides of (4.8). Hence, from (4.8) together with (4.10) we obtain

$$\begin{aligned} \log q^{1/2} \int_x^{x+1} \Big\{ & \log\big(\mathcal{O}_{-1,q}(t)^{-1}\mathcal{O}_q(t)^2\mathcal{O}_{1,q}(t)^{-1}\big) + p_q(t, q^t) \\ & - \frac{4}{(q^{1/2}-q^{-1/2})^2} \log \Gamma_q(t) \Big\}\, dt \\ = & -\frac{q^{x/2}+q^{-x/2}}{q^{1/2}-q^{-1/2}} [x]_q \log [x]_q. \end{aligned}$$

Here $p_q(x, y)$ is a polynomial of two variable which is computable explicitly but we omit the form.

Remark. In 1847, Kummer got the following formula for the gamma function:

$$\begin{aligned} \log \Gamma(x) = & \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(\log n)\sin(2\pi n x)}{n} + \frac{\log(2\pi)+\gamma}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} \\ & + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n} + \frac{1}{2}\log(2\pi). \end{aligned}$$

Hence the formula (4.5) is clear from this Fourier expansion. Also, from this expansion, it is obvious to see another formula like as

$$\int_0^1 \log \Gamma(t)\, \sin(2\pi n t)\, dt = \frac{\log 2\pi n + \gamma}{2\pi n} \quad (n = 1, 2, \ldots)$$

(see [6] for this sort of the calculations). In this sense, Kummer's type formulas for multiple gamma functions studied in [18] give us the same kind of formulas when the weights $\underline{\omega}$ are rational. Thus a general interesting question appears if one considers the non-commutative Fourier coefficients of the logarithm of the gamma function $\Gamma_{G,\rho}(x, \underline{\gamma})$ according to the irreducible decomposition of the representation of a subgroup of G.

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